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CONTACT PROBLEMS OF THE THEORY OF PLASTICITY FOR COMPLEX LOADING*

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The problem of the interaction between a stamp and an elastic-plastic body for a non-proportional change in the given loads and taking the contact area indeterminacy into account, is considered. It is assumed that the material properties are described by differential-linear or differential-non-linear relationships between the stress rates and the strain rates encompassing a fairly broad group of the theory of plasticity. It is shown that the initial problem in a generalized formulation is equivalent to a certain quasivariational inequality in the displacement velocities. By using a formulation in the form of a quasivariational inequality, existence conditions are studied for the solution, and a numerical method of investigation is proposed and verified.

Note that the method of integral equations, utilized extensively in the theory of elasticity, can be applied successfully only to certain special classes of contact problems of the theory of plasticity /1, 2/. Research on general questions of the theory of contact problems for elastic-plastic bodies refers to individual plasticity models /3—6/, and the numerical methods that have received extensive development are applied to contact problems under complex loading by using heuristic algorithms /7—9/ that require additional investigation and verification.

1. General formulation of the problem. We consider the quasistatic deformation of an elasto-plastic body occupying a domain Ω of a three-dimensional Cartesian space bounded by a piecewise-smooth surface Γ. The displacements and deformations are assumed to be small. We let t denote a monotonically increasing parameter associated with the loading process, which we shall call time. The solution of the problem is considered in at time interval $[0, T]$. We let $u_i(x, t)$, $\varepsilon_{ij}(x, t)$, and $\sigma_{ij}(x, t)$ denote the components of the displacement vector, and of the strain and stress tensors at the point $x = (x_1, x_2, x_3) \in \Omega$ at the time $t \in [0, T]$. We assume the body is in the unstressed and unstrained state at the initial time $t = 0$. We denote differentiation with respect to time by a point, and with respect to the space variables by a comma. The rule of summation over repeated subscripts is used.

It is assumed that the behaviour of the body material under complex laoding can be described by differential linear or differential non-linear relationships of the form

$$\sigma_{ij}^{\cdot} = A_{ijpq}(x, \varkappa_1, \varkappa_2, \ldots, \varkappa_r, \varepsilon_{\xi\eta}^{\cdot})\, \varepsilon_{pq}^{\cdot} \qquad (1.1)$$

The function A_{ijpq} is homogeneous of zeroth degree in $\varepsilon_{\xi\eta}^{\cdot}$ or generally independent of $\varepsilon_{\xi\eta}$ in the case of differential linear relationships. We take $\varkappa_1, \varkappa_2, \ldots, \varkappa_r$ to be values of certain functionals of the strain history. Relations for different versions of flow theory and for theories based on the slip concept /10/ can be represented in the form (1.1). Relationships (1.1) are a special version of the theory of elasto-plastic processes /11/. When $A_{ijpq} = A_{ijpq}(x)$ (1.1) correspond to linear elasticity theory for an inhomogeneous anisotropic body. Note that relationships of the form (1.1) can be used for both active loading and unloading processes.

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We introduce the function

$$W(x, \varkappa_1, \ldots, \varkappa_r, \dot{\varepsilon}_{\xi\eta}) = \frac{1}{2} A_{ijpq}(x, \varkappa_1, \ldots, \varkappa_r, \dot{\varepsilon}_{\xi\eta})\, \dot{\varepsilon}_{ij}\dot{\varepsilon}_{pq} \tag{1.2}$$

and we impose constraints on A_{ijpq} in such a way that the following conditions are satisfied:

a) $W(\ldots, \dot{\varepsilon}_{\xi\eta})$ is a convex, continuously differentiable function of $\dot{\varepsilon}_{\xi\eta}$;

b) There exist $\alpha_1 > 0, \alpha_2 > 0$ independent of the strain history such that the following inequalities are satisfied:

$$\alpha_1 \dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} \leqslant W(\ldots, \dot{\varepsilon}_{\xi\eta}) \leqslant \alpha_2 \dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} \tag{1.3}$$

The surface Γ consists of three parts $\Gamma_c, \Gamma_\sigma, \Gamma_u$. We will denote the normal to the surface Γ at the point x, external with respect to Ω, by $\nu(x)$.

A rigid stamp acts on the surface Γ_c, where the actual contact areas are not given in advance. It is assumed that there is no friction between the body surface Γ_c and the stamp surface. If the law of motion is given for the stamp as a rigid body, then the function $\Phi(x, t)$, whose values equal the distance between the point $x \in \Gamma_c$ and the surface of the stamp at the time t, can be constructed uniquely. the distance is measured along the direction of the normal $\nu(x)$. The function $\Phi(x, t)$ can also take negative values, which corresponds to insertion of the stamp. Since the body is in the unstrained state for $t = 0$ it must be required that $\Phi(x, 0) \geqslant 0,\ \forall x \in \Gamma_c$.

We let $u_\nu, u_\tau, \sigma_\nu, \sigma_\tau$ denote the normal and tangential components of the displacement and stress vectors at points of the surface Γ_c. Then interaction between the body and the stamp at points of the possible contact surface is described by the following conditions:

$$\begin{gathered}\sigma_\nu(x, t) \leqslant 0,\ \sigma_\tau(x, t) = 0,\ u_\nu(x, t) \leqslant \Phi(x, t) \\ \sigma_\nu(x, t)\,[u_\nu(x, t) - \Phi(x, t)] = 0,\ \forall x \in \Gamma_c,\ \forall t \in [0, T]\end{gathered} \tag{1.4}$$

The condition $u_\nu(x, t) \leqslant \Phi(x, t)$ expresses the requirement of mutual impenetrability of the body and the stamp, and the condition $\sigma_\nu(x, t) \leqslant 0$ denotes that there are no tensile stresses at points of the contact surface. We emphasize that the actual contact areas are not given in the formulation of conditions (1.4).

The forces are given on the surface Γ_σ

$$\sigma_{ij}(x, t)\,\nu_j(x) = S_i(x, t),\ \ S_i(x, 0) = 0,\ \ \forall x \in \Gamma_\sigma,\ \ \forall t \in [0, T] \tag{1.5}$$

and the displacements on the surface $\Gamma_u \neq \phi$

$$u_i(x, t) = U_i(x, t),\ U_i(x, 0) = 0,\ \forall x \in \Gamma_u,\ \ \forall t \in [0, T] \tag{1.6}$$

The body Ω is also subjected to the body forces

$$Q_i(x, t),\ Q_i(x, 0) = 0,\ \forall x \in \Omega,\ \ \forall t \in [0, T]$$

The initial problem is to determine the displacements $u_i(x, t)$, the strains $\varepsilon_{ij}(x, t)$, the stresses $\sigma_{ij}(x, t)$, satisfying the equilibrium equations, the Cauchy relationships, relationships (1.1), the boundary conditions (1.5) and (1.6), the conditions on the contact surface (1.4), and the initial conditions $u_i(x, 0) = \varepsilon_{ij}(x, 0) = \sigma_{ij}(x, 0) = 0,\ \forall x \in \Omega$.

2. Formulation in the form of quasivariational inequality. As before, we introduce certain functional spaces to determine the class of allowable functions. We understant $L^2(P)$ to be the Hilbert space of functions, square-summable in the manifold P; the scalar product in $L^2(P)$ is defined as follows:

$$(w_1, w_2) = \int_P w_1 w_2\, dP$$

We introduce the S.L. Sobolev space $H \equiv [W_2^{\,1}(\Omega)]^3$ of the vector functions $w(x) = (w_1(x), w_2(x), w_3(x))$ such that $w_i \in L^2(\Omega),\ w_{i,j} \in L^2(\Omega)$. We allot to H the structure of a Hilbert space by indicating the scalar product

$$(w^{(1)}, w^{(2)})_H = (w_i^{(1)}, w_i^{(2)})_{L^2(\Omega)} + (w_{i,j}^{(1)}, w_{i,j}^{(2)})_{L^2(\Omega)}$$

Finally, we define the Hilbert space $L^2(0, T; H)$ of the vector function $w(x, t)$ with the scalar product

$$(w^{(1)}, w^{(2)})_{L^2(0, T; H)} = \int_0^T (w^{(1)}, w^{(2)})_H\, dt$$

As is usual in Hilbert spaces, we understand the norm to be $\|w\| = \sqrt{(w, w)}$.

Henceforth we assume the displacement velocities $\dot{v}(x, t)$ to be elements of the space $L^2(0, T; H)$. We introduce the set V of admissible displacement velocities in which we include the velocities $\dot{v}(x, t)$ satisfying conditions (1.6) and the kinematic conditions from (1.4)

$$V = \left\{ v^\cdot \in L^2(0, T; H) \,\middle|\, \int_0^t v_i^\cdot(x, \tau)\, d\tau = U_i(x, t), \quad \forall x \in \Gamma_u, \right.$$
$$\left. \int_0^t v_\nu^\cdot(x, \tau)\, d\tau \leqslant \Phi(x, t), \quad \forall x \in \Gamma_c, \ \forall t \in [0, \ T] \right\}$$

We will obtain an analog of the principle of possible displacement velocities as it applies to contact problems with uncertain contact areas. We denote the actual displacement velocities in terms of $u_i^\cdot$, and those kinematically possible, in terms of $v_i^\cdot$.

We will define the possible strain rate tensor by the Cauchy relations

$$\zeta_{ij}^\cdot = {}^1/_2\,(v_{i,j}^\cdot + v_{j,i}^\cdot)$$

We understand $\varepsilon_{ij}^\cdot$, $\sigma_{ij}^\cdot$ to be components of the real strain rate tensors and the real stress rate tensors.

Applying Gauss's theorem to the identity

$$-\int_\Omega (\dot{\sigma}_{ij,j} + Q_i^\cdot)(v_i^\cdot - u_i^\cdot)\, d\Omega = 0, \quad \forall v^\cdot \in V$$

taking conditions (1.4)–(1.6) into account, we obtain the integral equation

$$\int_\Omega \sigma_{ij}^\cdot(\zeta_{ij}^\cdot - \varepsilon_{ij}^\cdot)\, d\Omega - \int_\Omega Q_i^\cdot(v_i^\cdot - u_i^\cdot)\, d\Omega - \int_{\Gamma_\sigma} S_i^\cdot(v_i^\cdot - u_i^\cdot)\, d\Gamma - \int_{\Gamma_c} \sigma_\nu^\cdot(v_\nu^\cdot - u_\nu^\cdot)\, d\Gamma = 0, \ \forall v^\cdot \in V, \ \forall t \in [0, T] \tag{2.1}$$

We introduce the subset $K(u)$ of vector-functions $v^\cdot(x, t)$ satisfying the additional condition

$$v_\nu(x, t) \equiv \int_0^t v_\nu^\cdot(x, \tau)\, d\tau = \Phi(x, t) \tag{2.2}$$

for all $(x, t) \in \Gamma_c \times [0, T]$ into the set V, such that

$$u_\nu(x, t) \equiv \int_0^t u_\nu^\cdot(x, \tau)\, d\tau = \Phi(x, t)$$

It can be shown that the equation

$$v_\nu^\cdot(x, t) = u_\nu^\cdot(x, t), \ \forall v^\cdot \in K(u) \tag{2.3}$$

holds at the actual contact points during the time of contact, with the exception of the initial contact and separation times.

At the contact-free points of Γ_c, $\sigma_\nu^\cdot = 0$ and therefore

$$\sigma_\nu^\cdot(v_\nu^\cdot - u_\nu^\cdot) = 0, \ \forall v^\cdot \in K(u), \text{ almost everywhere in } \Gamma_c \times [0, T] \tag{2.4}$$

We now examine (2.1) by considering $v^\cdot$ as an element of the set $K(u)$. Integrating in the interval $[0, T]$, and taking account of (2.4), we obtain

$$\int_0^T \left\{ \int_\Omega \sigma_{ij}^\cdot(\zeta_{ij}^\cdot - \varepsilon_{ij}^\cdot)\, d\Omega - \int_\Omega Q_i^\cdot(v_i^\cdot - u_i^\cdot)\, d\Omega - \int_{\Gamma_c} S_i^\cdot(v_i^\cdot - u_i^\cdot)\, d\Gamma \right\} dt = 0, \ \forall v^\cdot \in K(u) \tag{2.5}$$

We express $\sigma_{ij}^\cdot$ in terms of $\varepsilon_{ij}^\cdot$ according to (1.1) and we convert (2.5) to the form

$$J_1(u^\cdot, v^\cdot - u^\cdot) \equiv \int_0^T [a(\ldots, u^\cdot, v^\cdot - u^\cdot) - F(v^\cdot - u^\cdot)]\, dt = 0, \ \forall v^\cdot \in K(u) \tag{2.6}$$
$$a(\ldots, u^\cdot, v^\cdot) = \int_\Omega A_{ijpq}(\ldots, \varepsilon_{\xi\eta})\, \varepsilon_{ij}^\cdot \zeta_{pq}\, d\Omega$$
$$F(v^\cdot) = \int_\Omega Q_i^\cdot v_i^\cdot\, d\Omega + \int_{\Gamma_\sigma} S_i^\cdot v_i^\cdot\, d\Gamma$$

Relations (2.6) are analogous to a variational equation in the displacement velocities for classical boundary value problems and are distinguished by the fact that the allowable velocities $v^\cdot$ belong to the set $K(u)$ that depends on the desired solution.

We convert (2.6) to an equivalent form in such a way that the allowable velocities are elements of the set V. For $v^\cdot \in V \setminus K(u)$ the expression $J_1(u^\cdot, v^\cdot - u^\cdot)$ can take both positive and negative values. If components that take sufficiently large values to disturb the set $K(u)$ are appended to $J_1(u^\cdot, v^\cdot - u^\cdot)$ then for all $v^\cdot \in V$ the expression obtained will be non-negative. In particular, for all $v^\cdot \in V$ in $\varepsilon_0(v) > 0$ can be found such that

$$J_1(u^\cdot, v^\cdot - u^\cdot) + \int_0^T \int_{\Gamma_c} [\sigma_\nu(u)(v-u)/\varepsilon]\, d\Gamma\, dt \geqslant 0, \ \forall \varepsilon < \varepsilon_0(v) \tag{2.7}$$

We introduce the function $\psi(\sigma_\nu(u), v-u)$ as follows

$$\psi(\sigma_\nu(u), v-u) = \lim_{\varepsilon \to 0} \sigma_\nu(u)(v-u)/\varepsilon, \ \forall v^\cdot \in V$$

Obviously $\psi(\sigma_\nu(u), v-u) = +\infty$ if $\sigma_\nu(u) < 0$, $v < \Phi$, and $\psi(\sigma_\nu(u), v-u) = 0$ otherwise. Passing formally to the limit as $\varepsilon \to 0$, we obtain

$$J_2(\sigma_\nu(u), u^\cdot, v^\cdot - u^\cdot) \equiv J_1(u^\cdot, v^\cdot - u^\cdot) + \int_0^T F_c(\sigma_\nu(u), v-u)\, dt \geqslant 0, \ \forall v^\cdot \in V \tag{2.8}$$

$$F_c = \int_{\Gamma_c} \psi(\sigma_\nu(u), v-u)\, d\Gamma$$

If $\psi(\sigma_\nu(u), v-u) = +\infty$ in a set of non-zero measure in $\Gamma_c \times [0, T]$, then the inequality (2.8) is considered satisfied by definition.

Therefore, if $u^\cdot \in V$ is a solution of the problem in the initial (differential) formulation, then $u^\cdot \in V$ also satisfies the quasivariational inequality (2.8). The concept of the "quasivariational inequality" is utilized in connection with the fact that, unlike variational inequalities, the formulation of inequality (2.8) depends on the desired solution /3/.

Let us prove that the solution of the quasivariational inequality (2.8) is a generalized solution of the problem in the formulation of Sect.1. We will confine ourselves to conditions (1.4); the satisfaction of the remaining equations and conditions can be given a foundation exactly as in the case of contact problems for a non-linearly elastic body by formally replacing the displacements by velocities /4/. Let $u^\cdot(x, t) \in V$ be a solution of (2.8). We introduce the notation

$$\Gamma_c^{(1)}(t) = \{x \in \Gamma_c \mid u_\nu(x, t) = \Phi(x, t)\}, \quad \Gamma_c^{(2)}(t) = \Gamma_c \setminus \Gamma_c^{(1)}(t)$$

and we will show that

$$\sigma_\nu(x, t) \leqslant 0, \ \forall x \in \Gamma_c^{(1)}(t), \ \sigma_\nu(x, t) = 0, \ \forall x \in \Gamma_c^{(2)}(t).$$

In fact, if $\sigma_\nu(x, t) > 0$ for x forming a set of non-zero measure in $\Gamma_c^{(1)}(t)$, then for all $v^\cdot(x, t) \in V$ such that $v_\nu(x, t) < \Phi(x, t)$, $\forall x \in \Gamma_c^{(1)}(t)$, we have $F_c(\sigma_\nu(u), v-u) = -\infty$. If $\sigma_\nu(x, t) \neq 0$, $x \in \Gamma_c^{(2)}(t)$, then by selecting $v_i = u_i \pm \varphi_i$ on $\Gamma_c^{(2)}(t)$, we also arrive at a contradiction to the quasivariational inequality (2.8).

We finally formulate the results obtained in Sect.2.

Theorem 1. The solution of the initial problem formulated in Sect.1 satisfies the quasivariational inequality (2.8); conversely, solution (2.8) is a generalized solution of the problem in an initial (differential) formulation.

3. A method of solving the quasivariational inequality (2.8). We partition the segment $[0, T]$ into n parts by using $n+1$ nodes $t_0 = 0, t_1, \ldots, t_l, t_{l+1}, \ldots, t_n = T$; $t_l < t_{l+1}$. Since the velocities $v^\cdot(x, t)$ can as elements of the space $L^2(0, T; H)$, have discontinuities in time, we shall distinguish the times $t_l - 0$ preceding t_l and the times $t_l + 0$ following t_l. Within each interval (t_l, t_{l+1}) we approximate $v^\cdot(x, t)$ by linear functions $v_n^\cdot(x, t)$. The subscript n corresponds to the partition of the interval $[0, T]$ into n parts. From the requirements imposed on $v^\cdot \in V$ it follows that the nodal velocities $v_n^\cdot(x, t_l - 0)$ and $v_n^\cdot(x, t_l + 0)$ should satisfy the following conditions within the framewrok of the piecewise linear approximation being used:

$$\begin{aligned}
&v_{\nu n}(x, t_l - 0) \equiv v_{\nu n}(x, t_{l-1} + 0) + [v_{\nu n}(x, t_{l-1} + 0) + \\
&\quad v_{\nu n}^\cdot(x, t_l - 0)]\, \Delta t_l/2 \leqslant \Phi(x, t_l - 0), \ \forall x \in \Gamma_c, \ l = \\
&\quad 1, 2, \ldots, n, \ \Delta t_l = t_l - t_{l-1} \\
&v_{\nu n}^\cdot(x, t_l + 0) \leqslant \Phi^\cdot(x, t_l + 0) \text{ for all } x \in \Gamma_c, \text{ such that} \\
&v_{\nu n}(x, t_l - 0) = \Phi(x, t_l - 0), \ l = 0, 1, \ldots, n-1 \\
&v_i^\cdot(x, t_l - 0) = U_i^\cdot(x, t_l - 0), \ l = 1, 2, \ldots, n \\
&v_i^\cdot(x, t_l + 0) = U_i^\cdot(x, t_l + 0), \ l = 0, 1, \ldots, n-1
\end{aligned} \tag{3.1}$$

We let V_n denote the set of piecewise-linear functions $v_n^\cdot \in L^2(0, T; H)$ that satisfy (3.1). Let $u_n^\cdot \in V_n$ be the desired approximate solution, and $\sigma_{\nu n}$ the corresponding normal stress on Γ_c. Let $\sigma_{\nu n}(x, t) = \sigma_{\nu n}(x, t_{l-1} + 0)$, $\forall x \in \Gamma_c$, $\forall t \in (t_{l-1} + 0, t_l - 0)$ and we introduce the subset $K_n(u_n) \subset V$ of functions $v_n^\cdot \in V_n$ that satisfy the additional conditions

$$v_{\nu n}^\cdot(x, t_{l-1} + 0) = \Phi^\cdot(x, t_{l-1} + 0), \ v_{\nu n}(x, t_l - 0) = \Phi(x, t_l - 0) \tag{3.2}$$

for all $x \in \Gamma_c$ such that $\sigma_{\nu n}(x, t_{l-1} + 0) < 0$, $l = 1, 2, \ldots, n$.

We replace the integral in (2.8) by a sum using the trapezoidal formula. For $v_n^\cdot \in K_n(u_n)$ the values of ψ at the nodes $t_l \pm 0$ equal zero and there are no corresponding components in the integral sum. Let the approximate solution $u_n^\cdot(x, t) \in V_n$ already have been obtained for $t \leqslant t_l$. We will show how one can determine the nodal values $u_n^\cdot(x, t)$ for $t = t^* = t_l + 0$ and for $t = t^{**} = t_{l+1} - 0$. Let $\varkappa_{1n}^{(l)}, \ldots, \varkappa_{rn}^{(l)}$ denote values of the strain history functionals corresponding to the time $t_l - 0$. According to conditions (3.1) and (3.2), we introduce the set K_n^* of allowable velocities at the time t^* taking into account that for $t \leqslant t_l$ the approximate solution $u_n^{\cdot}$ has already been obtained

$$K_n^* = \{v_n^{\cdot}(x, t^*) \in H \mid v_{in}^{\cdot}(x, t^*) = U_i^{\cdot}(x, t^*),\ \forall x \in \Gamma_u,\ v_{\nu n}^{\cdot}(x, t^*) \leqslant \Phi^{\cdot}(x, t^*)$$

for all $x \in \Gamma_c$ such that

$$u_{\nu n}(x, t^*) = \Phi(x, t^*),\ v_{\nu n}^{\cdot}(x, t^*) = \Phi^{\cdot}(x, t^*)$$

for all $x \in \Gamma_c$ such that $\sigma_{\nu n}(x, t^*) < 0\}$.

We will require that the components in the integral sum corresponding to the time t^* should be non-negative, and we will obtain the following variational inequality to determine $u_n^{\cdot *} = u_n^{\cdot}(x, t^*)$:

$$a(x, \varkappa_{1n}^{(l)}, \ldots, \varkappa_{rn}^{(l)}, u_n^{\cdot *}, v_n^{\cdot *} - u_n^{\cdot *}) \geqslant F(t^*, v_n^{\cdot *} - u_n^{\cdot *}),\quad \forall v^{\cdot *} \in K_n^* \tag{3.3}$$

Considering $u_n^{\cdot}(x, t^*)$ known, we construct the set K_n^{**} of allowable velocities at the time t^{**}

$$K_n^{**} = \{v_n^{\cdot}(x, t^{**}) \in H \mid v_{in}^{\cdot}(x, t^{**}) = U_i^{\cdot}(x, t^{**}),\ \forall x \in \Gamma_u,$$
$$v_{\nu n}(x, t^{**}) \equiv u_{\nu n}(x, t^*) + [u_{\nu n}^{\cdot}(x, t^*) + v_{\nu n}^{\cdot}(x, t^{**})]\Delta t_{l+1}/2 \leqslant$$
$$\Phi(x, t^{**}),\ \forall x \in \Gamma_c, v_{\nu n}(x, t^{**}) = \Phi(x, t^{**}) \text{ for all } x \in \Gamma_c,$$
$$\text{such that } \sigma_{\nu n}(x, t^*) > 0\}$$

The values $u_n^{\cdot **} = u_n^{\cdot}(x, t^{**})$ are defined as the solution of the variational inequality

$$a(x, \varkappa_{1n}^{(l)}, \ldots, \varkappa_{rn}^{(l)}, u_n^{\cdot **}, v_n^{\cdot **} - u_n^{\cdot **}) \geqslant F(t^{**}, v_n^{\cdot **} - u_n^{\cdot **}), \forall v_n^{\cdot **} \in K_n^{**} \tag{3.4}$$

Conditions for solutions of the problems of the type (3.3) and (3.4) to exist are investigated fully in /3, 4, 12/. Using these conditions, we obtain that under the assumptions made about the function $W(\ldots, \varepsilon_{\xi\eta}^{\cdot})$ and under the following requirements on the functions $U_i^{\cdot}, \Phi_i^{\cdot}, Q_i^{\cdot}, S_i^{\cdot}$

$$U_i^{\cdot} \in H^{1/2}(\Gamma_u),\ \Phi^{\cdot} \in H^{1/2}(\Gamma_c),\ Q_i^{\cdot} \in L^2(\Omega),\ S_i^{\cdot} \in L^2(\Gamma_\sigma)$$

a unique solution of the problems (3.3) and (3.4) exists for almost all fixed $t \in [0, T]$.

A definition of the space $H^{1/2}(\ldots)$ is given in /3/. Simple and efficient algorithms /13/ are proposed for the numerical solution of problem of the type (3.3), (3.4).

4. The existence of a solution of the quasivariational inequality. The investigation of the convergence of the algorithm described in Sect.3 and the simultaneous existence of a solution of the quasivariational inequality (2.8) follows the idea of /14/ and consists of proving two assertions:

a) A subsequence, weakly convergent as $n \to \infty$, can be extracted from the sequence of approximate solutions $\{u_n^{\cdot}(x, t)\}$;

b) The limit of this subsequence satisfies the quasivariational inequality.

Let us prove the first assertion. Let $u_n^{\cdot *} = u_n^{\cdot}(x, t^*)$ be the solution of the variational inequality (3.3) and therefore

$$a(\ldots, u_n^{\cdot *}, v^{\cdot} - u_n^{\cdot *}) - F(t^*, v^{\cdot} - u_n^{\cdot *}) \geqslant 0,\ \forall v^{\cdot} \in K_n^* \tag{4.1}$$

We introduce the notation $w_n^{\cdot} = u_n^{\cdot *} - v$ and we convert (4.1) to the form

$$a(\ldots, w_n^{\cdot} + v^{\cdot}, w_n^{\cdot}) - F(t^*, w_n^{\cdot}) \leqslant 0,\ \forall v^{\cdot} \in K_n^* \tag{4.2}$$

It follows from condition (1.3) that for any $\alpha > 0$

$$a\left(\ldots, \frac{1}{\sqrt{2\alpha}} v^{\cdot} + \sqrt{\frac{\alpha}{2}} w_n^{\cdot}, \frac{1}{\sqrt{2\alpha}} v^{\cdot} + \sqrt{\frac{\alpha}{2}} w_n^{\cdot}\right) \geqslant 0$$

or

$$\frac{1}{2\alpha} a(\ldots, v^{\cdot}, v^{\cdot}) + \frac{\alpha}{2} a(\ldots, w_n^{\cdot}, w_n^{\cdot}) + a(\ldots v^{\cdot}, w_n^{\cdot}) \geqslant 0$$

Subtracting the last inequality from (4.2), we find that

$$\left(1 - \frac{\alpha}{2}\right) a(\ldots, w_n^{\cdot}, w_n^{\cdot}) \leqslant \frac{1}{2\alpha} a(\ldots, v^{\cdot}, v^{\cdot}) + F(t^*, w_n^{\cdot})$$

Using the Korn inequality /3/, we obtain the estimate

$$\alpha_1^* \| w_n^{\cdot} \|_H^2 \leqslant \frac{1}{2\alpha} a(\ldots, v^{\cdot}, v^{\cdot}) + F(t^*, w_n^{\cdot}) \tag{4.3}$$

We apply an analogous method in combination with the Sobolev /15/ imbedding theorems to estimate $F(t^*, w_n^{\cdot})$:

$$F(t^*, w_n^{\cdot}) \leqslant \frac{1}{2\alpha} \| Q^{\cdot} \|^2_{[L^2(\Omega)]^2} + \frac{\alpha\beta}{2} \| w_n^{\cdot} \|_H^2 + \frac{1}{2\alpha} \| S^{\cdot} \|^2_{[L^2(\Gamma_\sigma)]^2} + \frac{\alpha\gamma}{2} \| w_n^{\cdot} \|_H^2 \quad (4.4)$$

We note that because of the demands imposed on the functions $Q_i^{\cdot}$ and $S_i^{\cdot}$ in Sect.3, the corresponding norms in (4.4) are finite.

Combining (4.3) and (4.4), we obtain the final estimate

$$\| w_n^{\cdot} \|_H^2 \leqslant \beta_1 a(\ldots, v^{\cdot}, v^{\cdot}) + \beta_2 \| Q^{\cdot} \|^2_{[L^2(\Omega)]^2} + \beta_3 \| S^{\cdot} \|^2_{[L^2(\Gamma_\sigma)]^2}$$

The constants $\beta_1, \beta_2, \beta_3$ can be made positive because of the selection of arbitrary α. It hence follows that $\| w_n^{\cdot} \|^2 \leqslant C_1$, where C_1 is independent of the partition of the segment $[0, T]$. Since $u_n^{\cdot *} = w_n^{\cdot} + v^{\cdot}$, it can then be concluded that $\| u^{\cdot *} \|_H^2 < C_2$. It can similarly be shown that $\| u_n^{\cdot **} \|_H^2 < C_2$. Taking account of the method of determining $u_n^{\cdot}$ within the segment (t^*, t^{**}), we find that $\| u_n^{\cdot} \|_H^2 < C$, $C = \max(C_2, C_3)$, $\forall t \in [t^*, t^{**}]$, and therefore

$$\| u_n^{\cdot} \|^2_{L^2(0, T; H)} = \int_0^T \| u_n^{\cdot} \|_H^2 \, dt < CT$$

Hence, the approximate solution $u_n^{\cdot}$ is bounded in the norm $L^2(0, T; H)$ and consequently /15/ a subsequence that we denote by $\{u_n^{\cdot}\}$ and is weakly convergent to a certain element $u^{\cdot} \in L^2(0, T; H)$ can be extracted from the sequence $\{u_n^{\cdot}\}$.

We will show that $u^{\cdot}$ is a solution of the inequality (2.8). Using the partition of the interval $[0, T]$ introduced in Sect.3, we approximate $U_i^{\cdot}$, $\Phi^{\cdot}$ by piecewise linear functions $U_{in}^{\cdot}$, $\Phi_n^{\cdot}$ and $Q_i^{\cdot}$, $S_i^{\cdot}$ are piecewise-constant functions in time. All the approximations introduced converge strongly in the norm of the corresponding spaces /14/. If $v_n^{\cdot} \to v^{\cdot}$, $v_n^{\cdot} \in V_n$ as $n \to \infty$, then on the basis of the theorem on traces /3/, we conclude that $v^{\cdot} \in V$. Since the set V is convex and closed, and therefore, weakly closed, it can also be shown that $u^{\cdot} \in V$.

Let us substitute these approximations into the left side of (2.8); integrating with respect to the time, we obtain

$$\begin{aligned} &\frac{1}{2} \sum_{l=1}^{n} \{a(\ldots, u_n^{\cdot *}, v_n^{\cdot *} - u_n^{\cdot *}) + a(\ldots, u_n^{\cdot **}, v_n^{\cdot **} - u_n^{\cdot **}) - \\ &F(t^*, v_n^{\cdot *} - u_n^{\cdot *}) - F(t^{**}, v_n^{\cdot **} - u_n^{\cdot **}) + F_c(\sigma_{\nu n}^*, v_n^* - u_n^*) + \\ &F_c(\sigma_{\nu n}^*, v_n^{**} - u_n^{**}) + \frac{1}{3}[F_c(\sigma_{\nu n}^*, u_n^* - u_n^{**} - v_n^* - v_n^{**}) \Delta t_l + \\ &\int_\Omega A_{ijpq}(\ldots, \varepsilon_{\xi\eta n}^{\cdot *})(\varepsilon_{ijn}^{\cdot *} - \varepsilon_{ijn}^{\cdot **})(\zeta_{pqn}^{\cdot *} - \varepsilon_{pqn}^{\cdot *} + \zeta_{pqn}^{\cdot **} - \varepsilon_{pqn}^{\cdot **}) \, d\Omega \Big]\Big\} \Delta t_l \end{aligned} \quad (4.5)$$

The first four components are the sum of the left sides of the variational inequalities (3.3) and (3.4); this sum is non-negative in the case $v_n^{\cdot} \in K_n(u_n)$. If $v_n^{\cdot} \in V_n \setminus K_n(u_n)$, (4.5) takes the value $+\infty$. The components in square brackets tend to zero as $n \to \infty$.

We note that under the assumptions made about $W(\ldots, \varepsilon_{\xi\eta}^{\cdot})$, the functional $a(\ldots, v^{\cdot}, v^{\cdot})$ is weakly semi-continuous from below /16/. Taking account also of the strong convergence of the approximations introduced, and passing to the limit as $n \to \infty$, we obtain the required quasivariational inequality (2.8).

Therefore, the following has been proved:

Theorem 2. If the constitutive relationships of the form (1.1) are such that the function $W(\ldots, \varepsilon_{\xi\eta})$, constructed in conformity with (1.2), satisfies conditions a) and b) of Sect.1, while the selection of the functions $U_i^{\cdot}$, $\Phi^{\cdot}$, $Q_i^{\cdot}$, $S_i^{\cdot}$ is subject to the requirements

$$U_i^{\cdot} \in L^2(0, T; H^{1/2}(\Gamma_u)), \ \Phi^{\cdot} \in L^2(0, T; H^{1/2}(\Gamma_c))$$
$$Q_i^{\cdot} \in L^2(0, T; L^2(\Omega)), \ S_i^{\cdot} \in L^2(0, T; L^2(\Gamma_\sigma))$$

then the solution of the quasivariational inequality (2.8), or equivalently, the generalized solution of the initial problem $u^{\cdot} \in L^2(0, T; H)$ exists.

On the basis of the results obtained, an algorithm has been developed for the numerical solution of contact problems under complex loading in plane strain conditions. The variational inequalities (3.3) and (3.4) are replaced by equivalent problems of minimization /3, 4, 12/, whose discretization is realized by using a finite-element method. To solve the non-linear programming problems that occur, a generalization is used of the method of relaxation of vertices to the case of problems with constraints /17/. The set of programs developed in FORTRAN enables any theory of plasticity with governing relationships of the form (1.1) to be used.

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ON THE STEADY MOTION OF A CRACK WITH SLIP AND SEPARATION SECTIONS ALONG THE INTERFACE OF TWO ELASTIC MATERIALS*

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The pre-Rayleigh motion of a crack (slit) with a finite slip section adjoining the edge of the crack and a semi-infinite separation section along the line connecting two elastic materials is studied under the action of a moving load. The problem is first reduced to a Hilbert boundary value problem with three different singularities for a system of two analytic functions of a complex variable. Then, by using conformal mapping techniques, analytic continuation, and elimination of singularities it is reduced to a problem with two singularities that lends itself to splitting, and consequently, of solution in Cauchy-type integrals. The length of the slip section l is determined uniquely from additional physical conditions (no force of attraction on the slip section, and non-intersection of the slit edges in the separation zone) formulated in the form of inequalities. For a concentrated load at a distance L from the edge

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